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Effects of slip and uniform magnetic field on flow of immiscible couple stress fluids in a porous medium channel

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--***ABSTRACT***--
The paper deals with the flow of two immiscible couple stress fluids in a channel filled with porous medium in the presence of a transverse magnetic field. The flow in the channel is assumed to be governed by Stokes's couple stress fluid theory. At the fluid-fluid interface, we assume that the velocities, vorticities, shear stresses and couple stresses are continuous. The resulting equations are solved analytically by using slip conditions at the boundaries and interface conditions at the fluid-fluid interface. The fluid velocity components in both the regions are obtained in closed form. The effects of couple stress parameters, slip parameter, Hartmann number, porosity parameter on velocity components are investigated. In the absence of slip, the limiting cases are obtained and discussed briefly.
KEYWORDS;- *immiscible couple stress fluids, porous medium, Hartmann number, slip*

I. INTRODUCTION

Over the past five decades, the theory of couple stress fluids [1] has received considerable attention as the Newtonian fluid theory unable describe the characteristics of the fluids such as polymer fluids, liquid crystals, and suspension solutions. "The couple stress fluid theory is the simplest theory that shows all the important features and effects of couple stresses in a fluid medium. Couple stresses are a consequence of assuming that the mechanical action of one part of a body on another, across a surface, is equivalent to a force and a moment distribution. In classical nonpolar mechanics, moment distributions are not considered, and the mechanical action is assumed to be equivalent to a force distribution only. The main effect of couple stresses will be to introduce a length-dependent effect that is not present in the classical nonpolar theories"[2].

The flow of liquids through porous media plays a crucial role in the technological, natural and environmental processes for example, the hazardous waste spreading in soils, the crude oil displacement in petroleum engineering [3]. Various problems in the fields of reservoir mechanics and hydrodynamics involve two or three immiscible liquids of varying densities flowing through porous medium. Shail [4] studied the possibility of applying a two-fluid system to obtain the increased flow rates. Meyer and Garder [5] analyzed the flow of two/three immiscible fluids which are saturated in porous region by gravity. Bhattacharya [6] studied a time dependent pressure gradient driven flow of immiscible fluids between rigid plates.

Recently, several researchers studied the flows of two immiscible fluids, to mention few, such as Newtonian fluid –Newtonian fluid [7, 8], Newtonian fluid – couple stress fluid [9], and couple stress fluid – couple stress fluid [10], in channel configurations with the usual no-slip boundary conditions.

In general, the no-slip boundary condition is used when the fluid flows past rigid boundaries. However, it has been accepted now that a number of fluids such as polymerics and additive fluids slip/stick-slip on rgid boundaries. To describe the slip characteristics of fluid on the solid surface, "Navier introduced a more general boundary condition – that the fluid velocity component tangential to the solid surface, relative to the solid surface, is proportional to the shear stress on the fluid-solid interface" [11]. "Recent advances in the manufacture of micro-devices enable experimental investigation of fluid flow on the micro-scale, and many experimental results have provided evidence in support of the Navier slip condition" [12, 13].

To the extent the authors have surveyed, that the effects of slip as well as magnetic field on the flow of immiscible couple stress fluids in porous medium channel has not been studied. Hence, in this paper, flow of two immiscible couple stress fluids in a porous medium channel with a uniform applied magnetic field along with the slip at the boundaries is analyzed. The analytical expression for velocity components in both the regions are obtained in closed form. The effects of various parameters on the two velocity profiles are shown graphically and briefly discussed.

II. MATHEMATICAL FORMULATION

A fully developed laminar flow of two immiscible (i.e. non-soluble in each other) slightly conducting couple stress fluids in a porous medium between horizontal parallel plates is considered. The *x*-axis is taken along the fluid-fluid interface in the channel and the *y*-axis is taken perpendicular to the channel as shown in Fig. 1. The homogeneous, isotropic and rigid porous medium has porosity k. Let β_1 and β_2 be the slip coefficients of the fluids at lower and upper plates of the channel respectively. In both the regions, the flow is driven by a common pressure gradient. The fluid in the Region 1 ($-h \le y \le 0$), has density ρ_1, shear viscosity μ_1 and couple stress viscosity η_1. Similarly, the fluid in the Region 2 ($0 \le y \le h$), has density ρ_2, shear viscosity μ_2 and couple stress viscosity η_2. A uniform magnetic field B_o is assumed to be applied parallel to the positive *y*-direction. "The induced magnetic field is negligible in comparison to the applied magnetic field as the magnetic Reynolds number is less than unity" [14]. Except the Lorentz body force, no other body forces are acting on the fluids and there are no body couples.

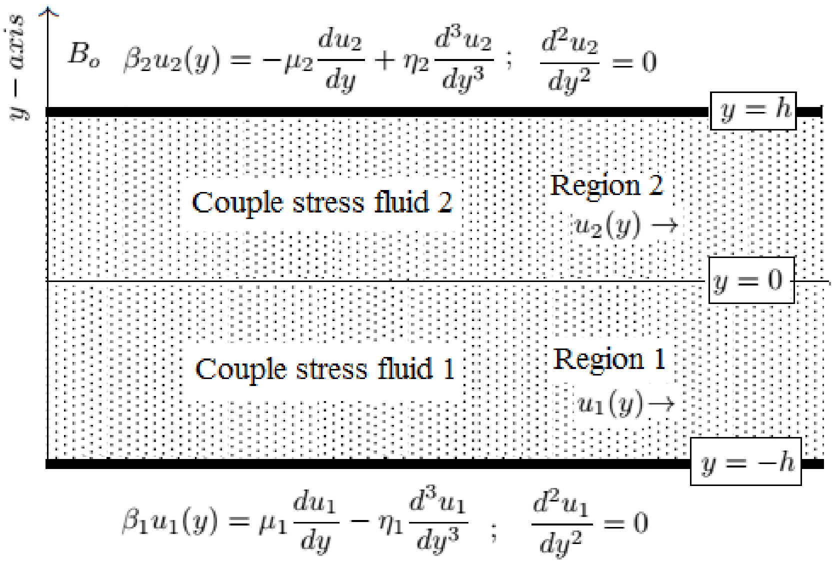

Fig.1. A schematic representation of flow of two immiscible couple stress fluids

The flow of couple stress fluid in the Region i (for $i = 1,2$) is governed by the following differential equation [1, 2]:

$$-\frac{dp}{dx} + \mu_i \frac{d^2u_i}{dy^2} - \eta_i \frac{d^4u_i}{dy^4} - \sigma_i B_o^2 u_i - \frac{\mu_i}{k} u_i = 0 \quad for\ i = 1, 2 \qquad (1)$$

where $-\frac{dp}{dx}$ represents the driving force i.e., the common pressure gradient , $\mu \frac{d^2u_i}{dy^2}$ represents the force due to shear stresses, $\eta \frac{d^4u_i}{dy^4}$ represents the force due to couple stresses, $\sigma_i B_o^2 u_i$ represent the Lorentz force due to applied magnetic field, $\frac{\mu_i}{k} u_i$ represents the retarding force due to porous media in both the fluid regions.

To determine the velocity components $u_1(y)$, $u_2(y)$ in Regions 1 and 2 described above, we take the following conditions:

(i) At the lower boundary of the channel (i.e., at $y = -h$):

$\beta_1 u_1 = \mu_1 \frac{du_1}{dy} - \eta_1 \frac{d^3u_1}{dy^3}$, (Slip condition) (2)

$\frac{d^2u_1}{dy^2} = 0$, (Vanishing of couple stresses) (3)

(ii) At the upper boundary of the channel (i.e., at $y = h$):

$\beta_2 u_2 = -\left(\mu_2 \frac{du_2}{dy} - \eta_2 \frac{d^3u_2}{dy^3}\right),$ (Slip condition) (4)

$\frac{d^2u_2}{dy^2} = 0,$ (Vanishing of couple stresses) (5)

(iii) At the Fluid –fluid Interface (i.e., at $y = 0$):

$u_1 = u_2$ (Continuity of velocities) (6)

$\frac{du_1}{dy} = \frac{du_2}{dy}$ (Continuity of vorticities) (7)

$\left(\mu_1 \frac{du_1}{dy} - \eta_1 \frac{d^3u_1}{dy^3}\right) = -\left(\mu_2 \frac{du_2}{dy} - \eta_2 \frac{d^3u_2}{dy^3}\right)$ (Continuity of shear stresses) (8)

$\eta_1 \frac{d^2u_1}{dy^2} = \eta_2 \frac{d^2u_2}{dy^2}$ (Continuity of couple stresses) (9)

By taking the following dimensionless quantities: $x^* = \frac{x}{h}$; $y^* = \frac{y}{h}$; $u_1^* = \frac{u_1}{U}$; $u_2^* = \frac{u_2}{U}$; $p^* = \frac{p}{\rho_1 U^2}$ we have the governing equations, boundary and interface conditions (in the non-dimensionalized form after dropping *'s) as

$$\frac{1}{s_1}\frac{d^4u_1}{dy^4} - \frac{d^2u_1}{dy^2} + (M^2 + \frac{1}{Da})u_1 = RP \ , \ \forall y \in [-1\ 0] \quad (10)$$

$$\frac{1}{s_2}\frac{d^4u_2}{dy^4} - \frac{d^2u_2}{dy^2} + \frac{1}{n_\mu}(n_\sigma M^2 + \frac{1}{Da})u_2 = \frac{1}{n_\mu}RP, \ \forall y \in [0\ 1] \quad (11)$$

(i) At the lower boundary of the channel (i.e., at $y = -1$):

$u_1 = \frac{1}{\alpha_1}\left(\frac{du_1}{dy} - \frac{1}{s_1}\frac{d^3u_1}{dy^3}\right),$ (Slip condition) (12)

$\frac{d^2u_1}{dy^2} = 0,$ (Vanishing of couple stresses) (13)

(ii) At the upper boundary of the channel (i.e., at $y = 1$):

$u_2 = -\frac{1}{\alpha_2}\left(\frac{du_2}{dy} - \frac{1}{s_2}\frac{d^3u_2}{dy^3}\right),$ (Slip condition) (14)

$\frac{d^2u_2}{dy^2} = 0,$ (Vanishing of couple stresses) (15)

(iii) At the Fluid –fluid Interface (i.e., at $y = 0$):

$u_1 = u_2$ (Continuity of velocities) (16)

$\frac{du_1}{dy} = \frac{du_2}{dy}$ (Continuity of vorticities) (17)

$\frac{du_1}{dy} - \frac{1}{s_1}\frac{d^3u_1}{dy^3} = n_\mu \left(\frac{du_2}{dy} - \frac{1}{s_2}\frac{d^3u_2}{dy^3}\right)$ (Continuity of shear stresses) (18)

$\frac{d^2u_1}{dy^2} = n_\mu \frac{s_1}{s_2}\frac{d^2u_2}{dy^2}$ (Continuity of couple stresses) (19)

where $P = -\frac{dp}{dx}$ is the constant pressure gradient, $R = \frac{\rho_1 U h}{\mu_1}$ is the Reynolds number, $\alpha_i = \frac{h\beta_i}{\mu_i}(i = 1,2)$ are the slip parameters, $s_i = \frac{\mu_i h^2}{\eta_i}$ $(i = 1,2)$ are the couple stress parameters, $M^2 = \frac{\sigma_1 {B_0}^2 h^2}{\mu_1}$ is the magnetic parameter (Hartmann number), $Da = \frac{k}{h^2}$ is the Darcy number (porosity parameter), $n_\sigma = \frac{\sigma_2}{\sigma_1}$ is the electric conductivity ratio, $n_\mu = \frac{\mu_2}{\mu_1}$ is the viscosity ratio and $\frac{\eta_2}{\eta_1}\left(= n_\mu \frac{s_1}{s_2}\right)$ is the ratio of coefficients of couple stress viscosities

III. SOLUTION OF THE PROBLEM

Region 1: $(-1 \le y \le 0)$

Solving Eqs. (10), and (11), we obtain the velocity $u_1(y)$ of Region 1 as

$$u_1(y) = C_{11}Cosh\,[\lambda_{11}y] + C_{12}Sinh\,[\lambda_{11}y] + C_{13}Cosh\,[\lambda_{12}y] + C_{14}Sinh\,[\lambda_{12}y] + \text{PI1} \quad (20)$$

Region 2: $(0 \le y \le 1)$

and for Region 2 is given by

$$u_2(y) = C_{21}Cosh\,[\lambda_{21}y] + C_{22}Sinh\,[\lambda_{21}y] + C_{23}Cosh\,[\lambda_{22}y] + C_{24}Sinh\,[\lambda_{22}y] + \mathrm{PI2} \qquad (21)$$

where $\lambda_{i1}, \lambda_{i2} = \sqrt{\frac{s_i}{2} \pm \frac{1}{2}\sqrt{s_i^2 - 4m_i}}$ for i = 1, 2;

$$m_1 = \left(M^2 + \frac{1}{Da}\right)s_1\,;\ \mathrm{PI1} = \frac{RPs_1}{m_1}\,;\ m_2 = \frac{1}{\eta_\mu}\left(\eta_\sigma M^2 + \frac{1}{Da}\right)s_2;\ \ \mathrm{PI2} = \frac{RPs_2}{\eta_\mu m_2};$$

The velocity components $u_1(y)$ and $u_2(y)$ involve eight constants $c_{i1},\ c_{i2},\ c_{i3},\ c_{i4}(i = 1,2)$. These c_{ij} are obtained from the interface and boundary conditions given by Eqs. (12) – (19) and the expressions for these constants are given in Appendix (A).

IV. DISCUSSION OF RESULTS

The steady flow of immiscible couple stress fluids in a channel saturated with homogenous porous medium under the influence of a uniform applied magnetic field is considered in the present study. The governing equations are analytically solved and closed form expressions for the velocities in both the regions are obtained. The effects of "pertinent parameters" on the velocities $u_i(y), (i = 1,2)$ are studied and presented through Figs. 2 -7. For the result analysis purpose, we consider that the parameters have fixed values as $M = 1,\ R = 2,\ P = 10,\ s_1 = 1,\ s_2 = 1,\ Da = 1,\ \alpha_1 = 10,\ \alpha_2 = 10,\ \eta_\mu = 1.1\,, \eta_\sigma = 1$. Further, if any variation in the parameters is indicated in that particular figure.

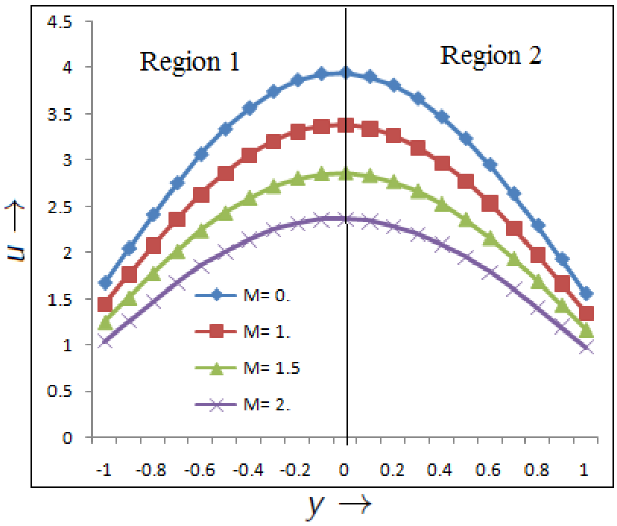

Fig.2. Effect of M on $u_i\ (i = 1,2)$

The effect of magnetic parameter on the distributions of velocity components is depicted in Fig. 2. The velocity profile attains the maximum for $M = 0$, this indicates the fluid flow in the absence of magnetic effects. As M is increasing, it is seen that the velocity components are decreasing. In the fluids flow, it signifies the retarding effect of the magnetic field.

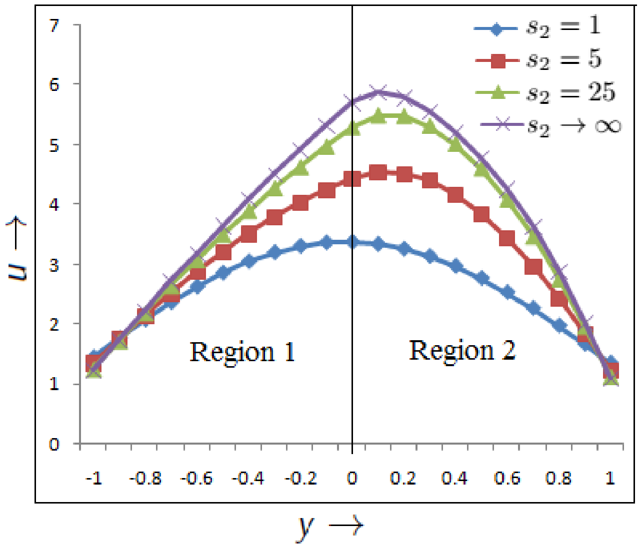

Fig.3. Effect of S_2 on u_i $(i = 1,2)$

From Fig.3, it is observed that as couple stress parameter s_2 is increasing, the velocity components are increasing. The increase in velocity is more pronounced in the region 2 and the velocity attained maximum value just above the fluid-fluid interface (at $y = 0.2$). As $s_2 \rightarrow \infty$ $(as\ \eta_2 \rightarrow 0)$, the velocity component $u_2(y)$ in the region 2 corresponds to the Newtonian fluid. This particular velocity profile represents the flow of immiscible couple stress and Newtonian fluids in a porous medium channel with slip boundary conditions.

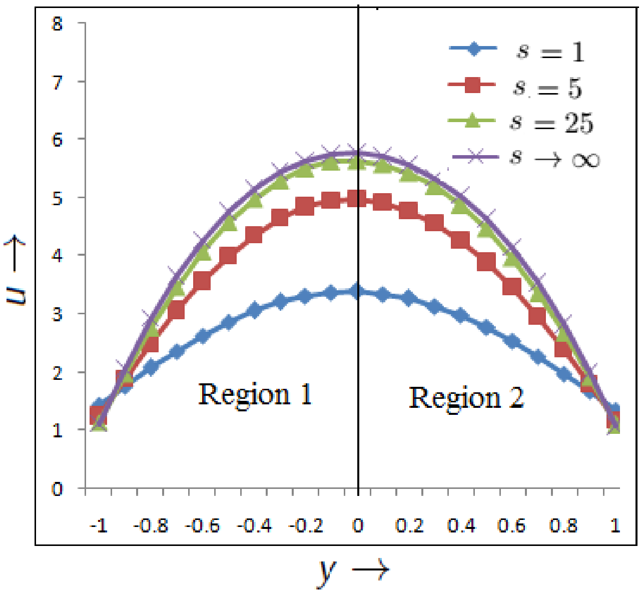

Fig.4. Effect of $s(= s_1 = s_2)$ on u_i $(i = 1,2)$

. In Fig. 4, the effects of couple-stress parameter s by assuming that both the fluids have same couple stresses$(i.e.,\ \eta_1 = \eta_2)$ is shown. In this case, along with $\eta_\mu = 1$, the two immiscible couple stress fluids treated as single couple stress fluid. The velocity is increasing, as s is increasing. Moreover, the increase in velocity is symmetric as we took the slip parameters $\alpha_1 = \alpha_2 = 10$ and velocity attained maximum value at the fluid-fluid interface (at $y = 0$). As $s \rightarrow \infty$ $(as\ \eta \rightarrow 0)$ indicates the flow of a Newtonian fluid in a porous medium channel with slip.

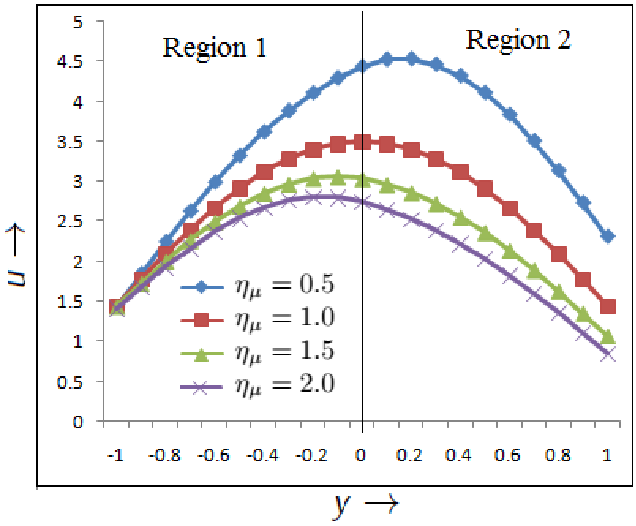

Fig.5. Effect of η_μ on u_i $(i = 1,2)$

The effect of the shear-viscosity ratio $\eta_\mu (= \frac{\mu_2}{\mu_1})$ is depicted in Fig. 5. The velocity is decreasing, as η_μ is increasing. For $\eta_\mu = 0.5$, the velocity in the region 2 is more than the velocity in the region 1. It confirms that the fluid with less viscosity has more velocity. For $\eta_\mu = 1$, the velocity components are symmetric about the interface.

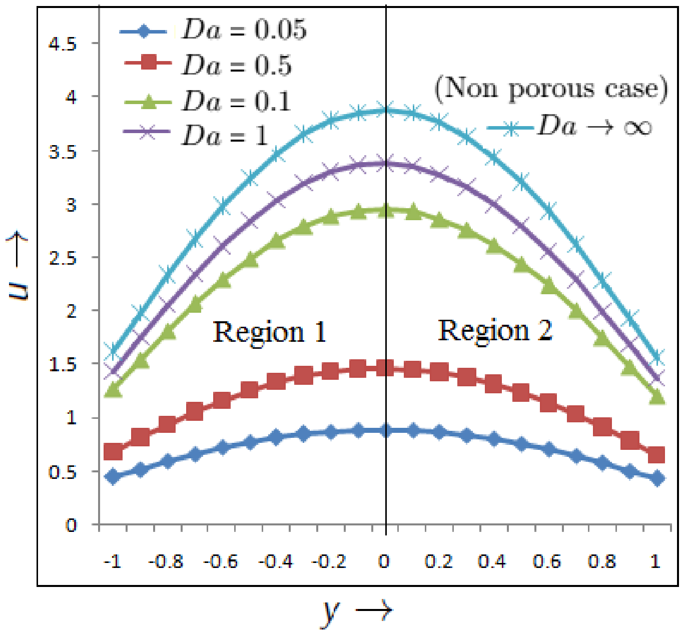

Fig.6. Effect of Da on u_i $(i = 1,2)$

. From Fig.6, it is observed that as the porosity parameter Da is increasing (i.e., as the permeability k is increasing), the velocity is increasing. As $Da \to \infty$, the velocity corresponds to the immiscible couple stress fluids flow in a channel in the absence of porous media.

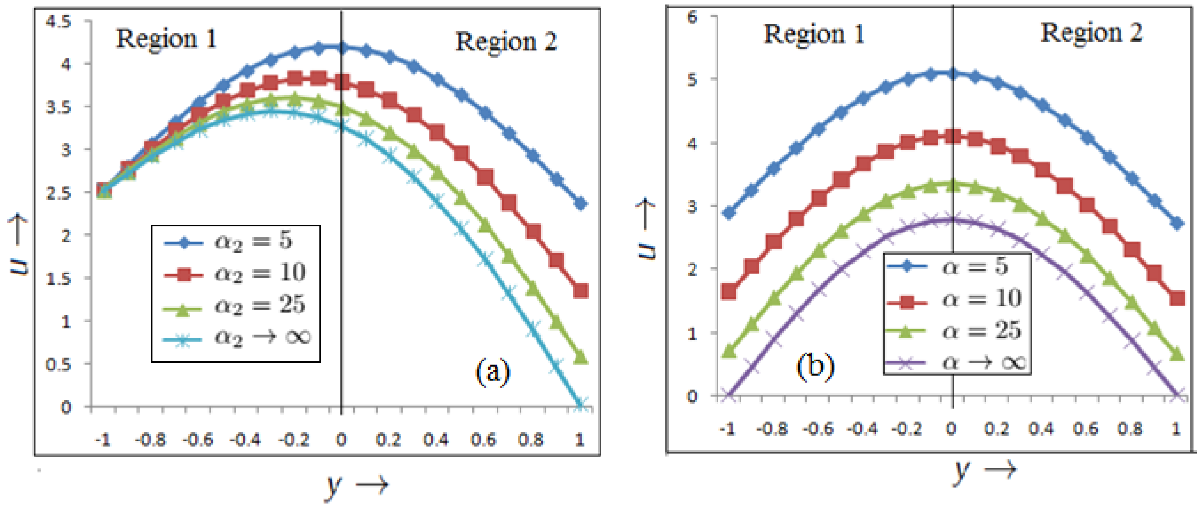

Fig.7. Effect of (a) α_2 and (b) $\alpha(=\alpha_1=\alpha_2)$ on u_i $(i=1,2)$

Fig.7(a) depicts the velocity variation with respect to the slip parameter α_2. The velocity is decreasing, as α_2 is increasing. The decrease is more near the upper plate in the region 2, as α_2 corresponds to the slip parameter at the upper plate. Fig.7(b) shows the velocity variation with respect to the slip parameter $\alpha(=\alpha_1=\alpha_2)$. The velocity is decreasing in both the regions, as α is increasing. Further, $\alpha \to \infty$ represents the flow with no slip.

V. CONCLUSIONS

Effects of slip and uniform magnetic field on the flow of two immiscible couple stress fluids in a porous medium channel are studied. The findings of the present study are listed as:

1. The velocity components in both the regions are decreased by the increase of slip and magnetic parameters.
2. The velocity components are increasing, as the parameters of couple stresses are increasing. The maximum velocities are attained as $s \to \infty$ $(\eta \to 0)$. This corresponds to the velocity components of immiscible Newtonian fluids flow in a porous media saturated channel.
3. The velocity is decreasing, as viscosity ratio is increasing. It confirms that the fluid with high viscosity has less velocity.
4. The velocity is increased by the increase of the porosity parameter Da. As $Da \to \infty$, the velocity is maximum in the absence of porous media.

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Appendix (A): C_{ij} ($i = 1, 2$ and $j = 1, 2, 3, 4$) expressions are given by

$C_{11} = (T_1 C_{12} - T_3 C_{13} + T_4 C_{14} - PI1)/T_1$; $C_{21} = -(T_6 C_{22} + T_7 C_{23} + T_8 C_{24}\ PI2)/T_5$;

$C_{12} = \left(G13\ C_{13} - G14 C_{14} - \frac{PI1 F_1}{T_1}\right)/G12$; $C_{22} = -\left(G57\ C_{23} + G58 C_{24} - \frac{PI2 F_5}{T_5}\right)/G56$;

$C_{13} = -(P3 + H1 C_{14} + H4 C_{23} + H3 C_{24})/H2$; $C_{23} = (P2 - G7 C_{14} - G8 C_{24} + G10)/G9$;

$C_{14} = (D5D4 - D6D2)/(D2D3 - D1D4)$; $C_{24} = (D1D6 - D3D5)/(D2D3 - D1D4)$;

$D1 = G1 - \frac{G3\ G7}{G9}$; $D2 = G2 - \frac{G3\ G8}{G9}$; $D3 = G4 - \frac{G5\ G7}{G9}$; $D4 = G6 - \frac{G5\ G8}{G9}$;
$D5 = \frac{G3\ P2}{G9} + \frac{G10\ G3}{G9} - \frac{G10\ L_{11}}{\lambda_{11}} - P1$; $D6 = \frac{G5\ P2}{G9} + \frac{G10\ G5}{G9} - \frac{H10\ P3}{H2} + P4 + PI1 - PI2$;

$G1 = \left(H5 - \frac{G13\ H1\ L_{11}}{G12\ H2}\right)$; $G2 = \left(H6 - \frac{G13\ H3\ L_{11}}{G12\ H2}\right)$; $G3 = \left(\frac{G57\ L_{21}\ n_\mu}{G56} - \frac{G13\ H4\ L_{11}}{G12\ H2}\right)$;
$G4 = \left(H7 - \frac{H1\ H10}{H2}\right)$; $G5 = \left(H8 - \frac{H10\ H4}{H2}\right)$; $G6 = \left(H9 - \frac{H10\ H3}{H2}\right)$; $G7 = \left(H11 - \frac{H13\ H1\ \lambda_{11}}{G12H2}\right)$;
$G8 = \left(H12 - \frac{H13\ H3\ \lambda_{11}}{G12H2}\right)$; $G9 = \left(\frac{G57\ \lambda_{21}}{G56} - \frac{H13\ H4\ \lambda_{11}}{G12H2}\right)$; $G10 = \frac{G13\ P3\lambda_{11}}{G12\ H2}$; $\eta_\eta = \frac{\eta_2}{\eta_1}\left(= n_\mu \frac{s_1}{s_2}\right)$;

$H1 = -\frac{G14\ T_2\ \lambda_{11}^2}{G12\ T_1} + \frac{T_4\ \lambda_{11}^2}{T_1}$; $H2 = \frac{G13\ T_2\ \lambda_{11}^2}{G12\ T_1} - \frac{T_3\ \lambda_{11}^2}{T_1} + \lambda_{12}^2$; $H3 = -\frac{G58\ T_6\ \eta_\eta\ \lambda_{21}^2}{G56\ T_5} + \frac{T_8\ \eta_\eta\ \lambda_{21}^2}{T_5}$;
$H4 = -\frac{G57\ T_6\ \eta_\eta\ \lambda_{21}^2}{G56\ T_5} + \frac{T_7\ \eta_\eta\ \lambda_{21}^2}{T_5} - \eta_\eta\ \lambda_{22}^2$; $H5 = -\frac{G14\ L_{11}}{G12} + L_{12}$; $H6 = \frac{G58\ L_{21} n_\mu}{G56} - L_{22} n_\mu$;
$H7 = -\frac{G14\ T_2}{G12 T_1} + \frac{T_4}{T_1}$; $H8 = -1 - \frac{G57\ T_6}{G56\ T_5} + \frac{T_7}{T_5}$; $H9 = -\frac{G58\ T_6}{G56\ T_5} + \frac{T_8}{T_5}$;
$H10 = 1 + \frac{G13\ T_2}{G12\ T_1} - \frac{T_3}{T_1}$; $H11 = -\frac{G14\ \lambda_{11}}{G12} + \lambda_{12}$; $H12 = \frac{G58\ \lambda_{21}}{G56} - \lambda_{22}$;
$P1 = \left(\frac{PI1 F_1 L_{11}}{G12 T_1} + \frac{PI2\ F_5\ L_{21} n_\mu}{G56\ T_5}\right)$; $P2 = \left(\frac{PI1\ F_1\ \lambda_{11}}{G12\ T_1} + \frac{PI2\ F_5\ \lambda_{21}}{G56\ T_5}\right)$;
$P3 = -\frac{PI1 \lambda_{11}^2}{T_1} + \frac{PI1\ F_1\ T_2 \lambda_{11}^2}{G12\ T_1^2} + \frac{PI2\ \eta_\eta \lambda_{21}^2}{T_5} + \frac{PI2\ F_5\ T_6\ \eta_\eta\ \lambda_{21}^2}{G56\ T_5^2}$;
$P4 = -\frac{PI1}{T_1} - \frac{PI1\ F_1\ T_2}{G12\ T_1^2} + \frac{PI2}{T_5} + \frac{PI2\ F_5\ T_6}{G56\ T_5^2}$;
$T_1 = \text{Cosh}[\lambda_{11}] + \frac{L_{11}}{a1}\text{Sinh}[\lambda_{11}]$; $T_2 = \text{Sinh}[\lambda_{11}] + \frac{L_{11}}{a1}\text{Cosh}[\lambda_{11}]$;
$T_3 = \text{Cosh}[\lambda_{12}] + \frac{L_{12}}{a1}\text{Sinh}[\lambda_{12}]$; $T_4 = \text{Sinh}[\lambda_{12}] + \frac{L_{12}}{a1}\text{Cosh}[\lambda_{12}]$;
$T_5 = \text{Cosh}[\lambda_{21}] + \frac{L_{21}}{a2}\text{Sinh}[\lambda_{21}]$; $T_6 = \text{Sinh}[\lambda_{21}] + \frac{L_{21}}{a2}\text{Cosh}[\lambda_{21}]$;
$T_7 = \text{Cosh}[\lambda_{22}] + \frac{L_{22}}{a2}\text{Sinh}[\lambda_{22}]$; $T_8 = \text{Sinh}[\lambda_{22}] + \frac{L_{22}}{a2}\text{Cosh}[\lambda_{22}]$;
$L_{ij} = \lambda_{ij}\left(1 - \frac{\lambda_{ij}^2}{s_i}\right)$ for $i, j = 1,2$; $Gij = \left(F_j - \frac{F_i T_j}{T_i}\right)$, for $ij = 12, 13, 14, 56, 57, 58$;
$F_1 = \lambda_{11}^2\ \text{Cosh}[\lambda_{11}]$; $F_2 = \lambda_{11}^2\ \text{Sinh}[\lambda_{11}]$; $F_3 = \lambda_{12}^2\ \text{Cosh}[\lambda_{12}]$; $F_4 = \lambda_{12}^2\ \text{Sinh}[\lambda_{12}]$;

$F_5 = \lambda_{21}^2\ \text{Cosh}[\lambda_{21}]$; $F_6 = \lambda_{21}^2\ \text{Sinh}[\lambda_{21}]$; $F_7 = \lambda_{22}^2\ \text{Cosh}[\lambda_{22}]$; $F_8 = \lambda_{22}^2\ \text{Sinh}[\lambda_{22}]$;